Problem 4.24

(a) Derive Equation 4.131 from Equation 4.130. *Hint:* Use a test function; otherwise you're likely to drop some terms.

(b) Derive Equation 4.132 from Equations 4.129 and 4.131. *Hint:* Use Equation 4.112.

Solution

Part (a)

The raising and lowering operators are given in Equation 4.130 on page 163.

$$\begin{cases} L_+ = +\hbar e^{+i\phi}\left(\dfrac{\partial}{\partial\theta} + i\cot\theta\dfrac{\partial}{\partial\phi}\right) \\ \\ L_- = -\hbar e^{-i\phi}\left(\dfrac{\partial}{\partial\theta} - i\cot\theta\dfrac{\partial}{\partial\phi}\right) \end{cases} \tag{4.130}$$

The aim is to prove Equation 4.131 on page 163.

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) \tag{4.131}$$

Let L_+L_- act on a test function and simplify the result.

$$\begin{aligned}
L_+L_-f &= \left[\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\right]\left[-\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right]f \\
&= -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(e^{-i\phi}\frac{\partial f}{\partial\theta} - ie^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right) \\
&= -\hbar^2 e^{i\phi}\left[\frac{\partial}{\partial\theta}\left(e^{-i\phi}\frac{\partial f}{\partial\theta} - ie^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right) + i\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\frac{\partial f}{\partial\theta} - ie^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right)\right] \\
&= -\hbar^2 e^{i\phi}\left[\left(e^{-i\phi}\frac{\partial^2 f}{\partial\theta^2} + ie^{-i\phi}\csc^2\theta\frac{\partial f}{\partial\phi} - ie^{-i\phi}\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}\right)\right. \\
&\qquad \left. + i\cot\theta\left(-ie^{-i\phi}\frac{\partial f}{\partial\theta} + e^{-i\phi}\frac{\partial^2 f}{\partial\phi\partial\theta} + i^2 e^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi} - ie^{-i\phi}\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right] \\
&= -\hbar^2 e^{i\phi}\left(e^{-i\phi}\frac{\partial^2 f}{\partial\theta^2} + ie^{-i\phi}\csc^2\theta\frac{\partial f}{\partial\phi} - \cancel{ie^{-i\phi}\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}}\right. \\
&\qquad \left. + e^{-i\phi}\cot\theta\frac{\partial f}{\partial\theta} + \cancel{ie^{-i\phi}\cot\theta\frac{\partial^2 f}{\partial\phi\partial\theta}} - ie^{-i\phi}\cot^2\theta\frac{\partial f}{\partial\phi} + e^{-i\phi}\cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= -\hbar^2\left[\frac{\partial^2 f}{\partial\theta^2} + i\underbrace{(\csc^2\theta - \cot^2\theta)}_{=1}\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned} L_+L_-f &= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2} + i\frac{\partial f}{\partial\phi}\right) \\ &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)f, \end{aligned}$$

which means

$$\boxed{L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right).}$$

Part (b)

Follow the hint and begin with Equation 4.112 on page 160.

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z \tag{4.112}$$

The aim is to prove Equation 4.132 on page 163.

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{4.132}$$

Let L^2 act on a test function and simplify the result.

$$\begin{aligned} L^2 f &= (L_+L_- + L_z^2 - \hbar L_z)f \\ &= L_+L_-f + L_z^2 f - \hbar L_z f \\ &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)f + \left(-i\hbar\frac{\partial}{\partial\phi}\right)\left(-i\hbar\frac{\partial}{\partial\phi}\right)f - \hbar\left(-i\hbar\frac{\partial}{\partial\phi}\right)f \\ &= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2} + i\frac{\partial f}{\partial\phi}\right) - \hbar^2\frac{\partial^2 f}{\partial\phi^2} + i\hbar^2\frac{\partial f}{\partial\phi} \\ &= -\hbar^2\frac{\partial^2 f}{\partial\theta^2} - \hbar^2\cot\theta\frac{\partial f}{\partial\theta} - \hbar^2\cot^2\theta\frac{\partial^2 f}{\partial\phi^2} - \cancel{i\hbar^2\frac{\partial f}{\partial\phi}} - \hbar^2\frac{\partial^2 f}{\partial\phi^2} + \cancel{i\hbar^2\frac{\partial f}{\partial\phi}} \\ &= -\hbar^2\frac{\partial^2 f}{\partial\theta^2} - \hbar^2\cot\theta\frac{\partial f}{\partial\theta} - \hbar^2(\cot^2\theta + 1)\frac{\partial^2 f}{\partial\phi^2} \\ &= -\hbar^2\frac{\partial^2 f}{\partial\theta^2} - \hbar^2\cot\theta\frac{\partial f}{\partial\theta} - \hbar^2\csc^2\theta\frac{\partial^2 f}{\partial\phi^2} \\ &= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right) \\ &= -\hbar^2\left[\frac{1}{\sin\theta}\left(\sin\theta\frac{\partial^2 f}{\partial\theta^2} + \cos\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right] \end{aligned}$$

Therefore,

$$\begin{aligned} L^2 f &= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\phi^2} \right] \\ &= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] f, \end{aligned}$$

which means

$$\boxed{L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right].}$$